

# On the zeta function on the line $\operatorname{Re}(s) = 1$ .

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## Abstract

We show the estimates

$$\inf_T \int_T^{T+\delta} |\zeta(1+it)|^{-1} dt = \frac{e^{-\gamma}}{4} \delta^2 + O(\delta^4), \quad (\delta > 0),$$

and

$$\inf_T \int_T^{T+\delta} |\zeta(1+it)| dt = \frac{\pi^2 e^{-\gamma}}{24} \delta^2 + O(\delta^4), \quad (\delta > 0),$$

as well as corresponding results for sup-norm,  $L^p$ -norm and other zeta-functions such as the Dirichlet  $L$ -functions and certain Rankin-Selberg  $L$ -functions. This improves on previous work of Balasubramanian and Ramachandra for small values of  $\delta$  and we remark that it implies that the zeta-function is not universal on the line  $\operatorname{Re}(s) = 1$ . We also use recent results of Holowinsky (for Maass wave forms) and Taylor et al. (Sato-Tate for holomorphic cusp forms) to prove lower bounds for the corresponding integral with the Riemann zeta-function replaced with Hecke  $L$ -functions and with  $\delta^2$  replaced by  $\delta^{11/12+\epsilon}$  and  $\delta^{8/(3\pi)+\epsilon}$  respectively.

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# 1 Introduction

## 1.1 Classical order and omega estimates

The importance of studying the Riemann zeta-function on the line  $\text{Re}(s) = 1$  was first realized by Von Mangoldt who proved in 1895 that  $\zeta(1 + it) \neq 0$  implies the prime number theorem. Hadamard [23] and de la Vallée-Poussin [17] shortly managed to prove this result independently. Since then the distribution of the zeta-values on the line  $\text{Re}(s) = 1$  has been studied by a lot of authors. For the general theory of the Riemann zeta-function, see for example the monographs of Ivić [29] and Titchmarsh [50]. Bohr [11] proved that the values  $\zeta(1 + it)$  are dense in  $\mathbb{C}$ . Assuming the Riemann hypothesis, Littlewood [38] showed that

$$\zeta(1 + it) \ll \log \log t, \quad \zeta(1 + it)^{-1} \ll \log \log t. \quad (1)$$

Bohr and Landau [12–14] proved the corresponding omega-estimates

$$\zeta(1 + it) = \Omega(\log \log t), \quad \zeta(1 + it)^{-1} = \Omega(\log \log t), \quad (2)$$

unconditionally, so Littlewood’s conditional bound should be the best possible. The best unconditional bound are the estimates

$$\zeta(1 + it) \ll (\log t)^{2/3}, \quad \zeta(1 + it)^{-1} \ll (\log t)^{2/3}(\log \log t)^{1/3},$$

of Vinogradov [51] and Korobov [33]. For recent improvements and the best known constants in these estimates see the paper of Granville-Soundararajan [22]. For related results in this direction, see also Hildebrink [24] and Lamzouri [34].

## 1.2 Universality

One interesting property for the Riemann zeta-function on lines  $\operatorname{Re}(s) = \sigma$  for  $1/2 < \sigma < 1$  is that of universality:

**Theorem 1.** *Let  $f$  be a continuous function on the interval  $[0, H]$  and let  $1/2 < \sigma < 1$ . Then for any  $\epsilon > 0$  there exists a  $T$  such that*

$$\max_{t \in [0, H]} |f(t) - \zeta(\sigma + iT + it)| < \epsilon.$$

This is a simple consequence of the Universality theorem of Bagchi [8] (see also Steuding [47] or Laurinćikas [35]) which generalizes the classical Universality result of Voronin [54, 55]. This version of universality is proved in [3, Corollary 2], where the requirement that  $f(t)$  is nonvanishing on the interval that follows from a trivial application of Bagchi's theorem is removed.

What about this theorem on the lines  $\sigma = 1/2$  or  $\operatorname{Re}(s) = 1$ ? A related result of Voronin [53] that predates his universality result is the following (For a discussion of on how these result are related, see [4]):

**Theorem 2.** *Suppose that  $1/2 < \sigma \leq 1$ . Then the set of  $n$ -tuples*

$$\{(\zeta(\sigma + it), \zeta'(\sigma + it), \dots, \zeta^{(n)}(\sigma + it)) : t \in \mathbb{R}\}$$

*is dense in  $\mathbb{C}^n$ .*

For  $\sigma = 1/2$  this was recently proved to be false by Garunkštis and Steuding [21, Theorem 1. (ii)]. Thus it is not surprising that we manage to show that Theorem 1 is also false on  $\sigma = 1/2$ , although it does not immediately follow from this result. We use a similar argument as that of Garunkštis and Steuding. It follows that the Riemann zeta-function is not universal even in  $L^1$  or  $L^2$ -norm, by the simple fact that the Hardy  $Z$ -function is real and hence the argument of the Riemann zeta-function (up to  $\pm 1$ ) on the critical line is determined by the Gamma-factors of the functional equation. Stirling's formula implies that the Gamma-factors are too regular to allow for universality. For a thorough discussion and a detailed proof of this result see our paper [4].

Since Theorem 2 is true for  $\sigma = 1$  we might guess that Theorem 1 should be true as well for  $\sigma = 1$ . In view of this it might seem surprising that Theorem 1 is infact false on this line. The proof is rather simple. Given large enough  $H$  it follows indirectly from the method in [2] that Theorem 1 can not be valid for  $\sigma = 1$ , since otherwise our proof method which used universality on lines would have worked to disprove a case known to be true.

## 1.3 The Balasubramanian Ramachandra method

More directly, the method of Balasubramanian and Ramachandra gives the following lower estimate [41]

$$\int_T^{T+H} |\zeta(1 + it)| dt \geq C_0 H, \quad H \geq H_0, \quad (3)$$

for some absolute constants  $C_0, H_0 > 0$  not depending on  $T$ . It is clear that Theorem 1 is not true (not even in  $L^1$ -norm) for  $H = H_0$  and  $\sigma = 1$  since if  $\delta = C_0 H$ , then  $\zeta(1 + it + iT)$  can clearly not approximate any function  $f \in C(0, H)$  with  $L^1$  norm less than  $\delta$ .

The results of Balasubramanian and Ramachandra are very strong and satisfying in some ways. For example their method gives the same omega-estimates for  $|\zeta(1 + it)|$  and  $|\zeta(1 + it)|^{-1}$  as (2) even in short intervals  $t \in [T, T + H]$  if  $H = T^\theta$  with  $0 < \theta < 1$ . This follows from the result

$$\max_{T \leq t \leq T+H} |\zeta(1 + it)| \gg \log \log H, \quad H \geq H_0. \quad (4)$$

Assuming the Riemann hypothesis, when  $H = T^\theta$  for  $0 < \theta < 1$  this is the best possible result in these intervals up to a constant depending on  $\theta$ , by Littlewood's result (1). The possibly exceptions where (1) is not true can however be shown to be rather sparse unconditionally. By replacing the Riemann hypothesis with known density theorems [29, Chapter 11], we can show that the measure of the exceptional set

$$\text{meas} \{0 \leq t \leq T : \text{Eq. (1) is false} \} \ll_\epsilon T^\epsilon$$

is small for each  $\epsilon > 0$ . This follows from the fact that the set  $\{z = \sigma + it : 1 - \epsilon/4 \leq \sigma, T_0 - T_0^{\epsilon/4} \leq t \leq T_0 + T_0^{\epsilon/4}\}$  is a zero free region with at most a measure of  $T^\epsilon$  exceptions for  $0 \leq T_0 < T$ . Whenever we have such a zero-free region around  $\zeta(1 + iT_0)$ , Littlewood's method [38] applies and the logarithm of the zeta-function can be estimated by Dirichlet polynomial of length some power of  $\log T$ .

This implies that Balasubramanian-Ramachandra's result (4) is the best possible (up to a constant) on average in  $T$  even if the Riemann hypothesis is assumed to be false. That is we have that

$$\max_{T \leq t \leq T+H} |\zeta(1 + it)| \ll \log \log T, \quad H \ll T_0^\theta. \quad (5)$$

for  $0 \leq T \leq T_0$ , with at most a measure of  $T_0^{\theta+\epsilon}$  exceptions.

That it gives the conjectured right order of magnitude is typical of the method of Balasubramanian and Ramachandra (see Ramachandra's monograph [42]). Similarly on the critical line it will give the same lower bound for higher moments of the Riemann zeta-function in short intervals  $[T, T + H]$  for  $H = T^\theta$  as the conjectured upper bound. In the critical strip it is required that

$$H \gg \log \log T.$$

in order for Balasubramanian-Ramachandra's method to work, which is weaker than  $H \gg 1$ , Eq. (4) on the line  $\text{Re}(s) = 1$ . We will discuss the limits of this method in [5].

## 1.4 A new lower bound in short intervals

While it follows from Balasubramanian-Ramachandra's method that the zeta-function is not universal on the line  $\text{Re}(s) = 1$  for functions  $f \in C(0, H_0)$ , it does

not rule out the existence of some small  $0 < \delta < H_0$  such that each continuous function  $f(t)$  on the interval  $[0, \delta]$  can be approximated by  $\zeta(1 + iT + it)$ . In this paper we will devise new methods that works on  $\text{Re}(s) = 1$  for arbitrarily short intervals. Our main result will be the following theorem:

**Theorem 3.** *We have the following estimates for the  $L^1$  norm of the zeta-function and its inverse in short intervals:*

$$\begin{aligned} (i) \quad & \inf_T \int_T^{T+\delta} |\zeta(1 + it)| dt = \frac{\pi^2 e^{-\gamma}}{24} \delta^2 + O(\delta^4), \\ (ii) \quad & \inf_T \int_T^{T+\delta} |\zeta(1 + it)|^{-1} dt = \frac{e^{-\gamma}}{4} \delta^2 + O(\delta^4), \end{aligned}$$

for  $\delta > 0$ . Furthermore, both estimates are valid if  $\inf_T$  is replaced by  $\liminf_{T \rightarrow \infty}$ .

We also have the corresponding theorem when we consider the half-plane  $\text{Re}(s) > 1$ . This will in fact have a slightly simpler proof.

**Theorem 4.** *We have the following estimates for the  $L^1$  norm of the zeta-function and its inverse in short intervals:*

$$\begin{aligned} (i) \quad & \inf_{\sigma > 1} \int_T^{T+\delta} |\zeta(\sigma + it)| dt = \frac{\pi^2 e^{-\gamma}}{24} \delta^2 + O(\delta^4), \\ (ii) \quad & \inf_{\sigma > 1} \int_T^{T+\delta} |\zeta(\sigma + it)|^{-1} dt = \frac{e^{-\gamma}}{4} \delta^2 + O(\delta^4), \end{aligned}$$

for  $\delta > 0$ . Furthermore, both estimates are valid if  $\inf_T$  is replaced by  $\liminf_{T \rightarrow \infty}$ .

### 1.5 On which lines is the zeta-function universal?

As a consequence of Theorem 3 it is clear that  $\zeta(s)$  can not be universal on the line  $\text{Re}(s) = 1$  even for short intervals. For whenever we have that

$$\int_0^\delta |f(t)| dt < \frac{\pi^2 e^{-\gamma}}{24} \delta^2 + O(\delta^4), \quad \text{or} \quad \int_0^\delta |f(t)|^{-1} dt < \frac{e^{-\gamma}}{4} \delta^2 + O(\delta^4),$$

then  $\zeta(1 + it + iT)$  can not approximate the function  $f(t)$  arbitrarily closely even in  $L^1$  norm, and certainly not in sup-norm.

By combining Theorem 1 with the observations that the zeta function is not universal on the lines  $\text{Re}(s) = 1/2$  and  $\text{Re}(s) = 1$  we obtain the following result:

**Theorem 5.** *Under the assumption of the Riemann hypothesis we have that the only lines where the Riemann zeta-function is universal in  $L^1$ -norm and for some interval  $[0, \delta]$  are the lines  $\text{Re}(s) = \sigma$  for  $1/2 < \sigma < 1$ . Furthermore for those lines we have universality for any interval  $[0, H]$  and in sup-norm.*

*Proof.* The only difficult remaining lines to consider are the lines  $\operatorname{Re}(s) = \sigma$  for  $0 < \sigma < 1/2$ . Here we will need the Riemann hypothesis. From the Riemann hypothesis it follows that  $\log \zeta(\sigma + it) \ll (\log t)^{2-2\sigma+\epsilon}$  [50, Theorem 14.2], whenever  $1/2 \leq \sigma \leq 1$ . From this we see that

$$\int_T^{T+\delta} |\zeta(\sigma + it)| dt \gg_\epsilon T^{-\epsilon}, \quad (1/2 < \sigma < 1). \quad (6)$$

By combining (6) with the functional equation and Stirling's formula for the Gamma-factors we obtain that

$$\int_T^{T+\delta} |\zeta(\sigma + it)| dt \gg_\epsilon T^{1/2-\sigma-\epsilon}, \quad (0 < \sigma < 1/2).$$

Since this will tend to infinity as  $T \rightarrow \infty$  we see that we do not have universality in  $L^1$ -norm on the line  $\operatorname{Re}(s) = \sigma$  for  $0 < \sigma < 1/2$ . That we do not have universality on the lines  $\operatorname{Re}(s) = \sigma$  with  $\sigma \leq 0$  and the lines that are not parallel to the imaginary axis follows trivially from the functional equation and the definition of the Riemann zeta-function.  $\square$

**Problem 1.** Prove Theorem 5 unconditionally.

We remark that it would be sufficient to prove eq. (6) unconditionally. This however seems quite difficult. In our paper [4] we use some convexity estimates from Ramachandra's book [42] to prove (6) under the Lindelöf hypothesis and thus we have managed to relax the condition of the Riemann hypothesis somewhat.

Results like (6) would have other important applications also. For example Ivić [28] showed that good lower estimates for this integral (sufficiently explicit in  $\delta$ ) have applications on the problem of estimating the multiplicities of the zeta-zeroes.

Possibly another idea can be useful to attack Problem 1?

## 2 Some approaches to Theorem 3

We will first show some approaches to Theorem 3, that although they will not obtain the full strength of Theorem 3, will yield non trivial results, for example sufficient to prove non universality on the line  $\operatorname{Re}(s) = 1$ . Although the results are superseded, the ideas might still have some interest. If nothing else they describe how our original approaches to this problem have improved with time. For the reader mainly interested in our final proof this section can be skipped.

### 2.1 Lower bound - The Mollifier method

We will first sketch how to prove the lower bound in Theorem 3 (i) with  $\delta^2$  is replaced by  $\delta^{2+\epsilon}$ . The lower bound in Theorem 3 (ii) can be proved similarly. This gave us our first of the non universality of the Riemann zeta-function on its abscissa of convergence, and it was first presented at the Zeta Function Days in

Seoul, September 1st - 5th, 2009. We will find a new use of this method in [5]. Introduce the standard Mollifier<sup>1</sup>:

$$M_X(s) = \sum_{1 \leq n < X} \mu(n) n^{-s}.$$

where  $X = e^{\delta^{-1-\epsilon}}$ . Consider the integral

$$(*) = \int_{-\infty}^{\infty} \phi((t-T)/\delta) \zeta(1+it) M_X(1+it) dt,$$

for a test function  $\phi \in C_0^\infty(\mathbb{R})$ , with support on  $[0, \delta]$  such that  $2\pi\hat{\phi}(0) = 1$ . By the triangle inequality we obtain that

$$\int_T^{T+\delta} |\zeta(1+it)| dt \gg \frac{(*)}{\max_{t \in [T, T+\delta]} |M_X(1+it)|}.$$

The lower bound in Theorem 3 (i) with  $\delta^2$  replaced by  $\delta^{2+\epsilon}$  follows by the estimates

$$|M_X(1+it)| \ll \log X = \delta^{-1-\epsilon}, \quad \text{and} \quad (*) = \delta + O(\delta^{1+\epsilon}).$$

Here the first estimate comes from estimating the Dirichlet polynomial  $M_X(1+it)$  trivially by its absolute values, and the second estimate follows from the fact that  $\hat{\phi}(x) \ll_N x^{-N}$  as  $x \rightarrow \infty$  for each  $N > 0$  (Schwartz class maps to Schwartz class under Fourier transforms). By choosing a somewhat smaller value of  $X$  and by using theorems of Paley and Wiener (see e.g. [6, Corrolary 2]) related to quasianalyticity on how fast Fourier-transforms of functions with compact support can go to zero, we can obtain improvements of this result. For example it can be shown that  $\delta^{2+\epsilon}$  may be replaced by

$$\delta^2 |\log \delta|^{-1-\epsilon}, \quad \text{or} \quad \delta^2 |\log \delta|^{-1} (|\log |\log \delta||)^{-1-\epsilon}.$$

However, the same Paley-Wiener theorem will also give limits for how good estimates this method can yield. For example, this method will not be able to yield the bounds

$$\delta^2 |\log \delta|^{-1}, \quad \text{or} \quad \delta^2 |\log \delta|^{-1} (|\log |\log \delta||)^{-1}.$$

Hence this method of proof is not strong enough to obtain Theorem 3.

## 2.2 Upper bounds

It is rather easy to see that the lower estimate in Theorem 3 (ii)<sup>2</sup>, can not be improved to anything better than something of the order of  $\delta^2$ . This follows from the example

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<sup>1</sup>This has also been used by Selberg [45] to show a positive proportion of zeros on the critical line and it has also has important applications for zero density estimates (See [29], chapter 11).

<sup>2</sup>A similar example (although slightly more complicated) can be given for Theorem 3 (i).

**Example 1.** We have that

$$\int_0^\delta |\zeta(1+it-\delta/2)|^{-1} dt = \frac{\delta^2}{4} + O(\delta^4).$$

*Proof.* From the Taylor expansion of the Riemann zeta function at  $\text{Re}(s) = 1$  it follows that

$$\frac{1}{\zeta(s)} = s - 1 - \gamma(s-1)^2 + O((s-1)^3).$$

This implies that

$$\begin{aligned} \int_0^\delta |\zeta(1+it-\delta/2)|^{-1} dt &= \int_0^\delta |(t-\delta/2) + i\gamma(t-\delta/2)^2 + O((t-\delta/2)^3)| dt \\ &= \frac{\delta^2}{4} + O(\delta^4). \end{aligned}$$

□

One may ask if this counterexample the best possible? In fact at the Zeta-Function Days in Seoul we asked the following question:

**Question.** (Asked at the ZFD in Seoul) Suppose that  $A(s)$  is a Dirichlet series such that its coefficients and the coefficients of its inverse are absolutely bounded by 1. Is it true that

$$\int_0^\delta |A(1+it)| dt \geq \int_{-\delta/2}^{\delta/2} |\zeta(1+it)|^{-1} dt,$$

with equality iff  $A(s) = e^{i\theta} \zeta(s - i\delta/2)^{-1}$ ?

*Answer:* No

Within a month of posing the question we found some other examples that give better estimates:

**Example 2.** If

$$A(s) = (\zeta(s)^2 \zeta(s + i\delta/6) \zeta(s - i\delta/6))^{-1/4}.$$

Then

$$\int_{-\delta/2}^{\delta/2} |A(1+it)| dt \leq 0.9518 \int_{-\delta/2}^{\delta/2} |\zeta(1+it)|^{-1} dt$$

for sufficiently small  $\delta$ .

This follows from the integral

$$\int_{-1/2}^{1/2} |t^2 - 1/6|^{1/4} |t|^{1/2} dt < \frac{0.9518}{4}.$$

These examples are still not as good as what is possible, since

$$e^{-\gamma} = 0.5615 < 0.9518 < 1,$$

they do not give as good results as Theorem 3, which follows from a different construction that gives us something very close to the optimal result.



### 3 The logarithmic $L^1$ -norm

#### 3.1 Jensen's inequality

We use the following version of Jensen's inequality

$$\frac{1}{\delta} \int_T^{T+\delta} \log |\zeta(1+it)| dt \leq \log \left( \frac{1}{\delta} \int_T^{T+\delta} |\zeta(1+it)| dt \right). \quad (7)$$

This version of the Jensen's inequality can be obtained from the inequality between the arithmetic and geometric means of  $N$  points, by letting  $N \rightarrow \infty$ , taking the logarithm of both sides and interpreting the sums as Riemann sums. In general we can replace the logarithm function with any concave function. This inequality has been applied to the zeta-function before, and can be found for example in Titchmarsh [50], equation 2 on page 230. However it does not seem as anyone applied the inequality on this particular problem before. Theorem 3 thus reduces to the problem of estimating the integral of the logarithm of the zeta-functions in short intervals.

#### 3.2 The logarithmic $L^1$ norm of Dirichlet series with multiplicative coefficients

Jensen's inequality suggests that we should study the logarithm of the Riemann zeta-function. This will in fact be much easier, since we can integrate the logarithm of the Riemann zeta-function term-wise and we have better convergence properties.

**Proposition 1.** *Let  $\mathcal{M} = \{A(s) = \sum_{n=1}^{\infty} a_n n^{-s}\}$  be the set of Dirichlet series, with completely multiplicative coefficients  $a_{nm} = a_n a_m$ , such that  $|a_n| \leq 1$ . Then we have for  $\sigma \geq 1$  that*

$$\inf_{A \in \mathcal{M}} \int_{-\delta/2}^{\delta/2} \log |A(\sigma + it)|^{-1} dt, \quad \text{and} \quad \inf_{A \in \mathcal{M}} \int_{-\delta/2}^{\delta/2} \log |A(\sigma + it)| dt$$

are minimized by  $A(s) = \zeta_{\delta}(s)$ , and  $A(s) = \zeta(2s)/\zeta_{\delta}(s)$  respectively, where

$$\zeta_{\delta}(s) = \prod_{p \text{ prime}} (1 - \varepsilon_p p^{-s})^{-1}, \quad \varepsilon_p = \varepsilon(\delta \log p).$$

and  $\varepsilon(x) = \text{sign}(\sin \frac{x}{2}) = (-1)^{\lfloor x/(2\pi) \rfloor}$ .

This follows from the Euler product and the following Lemma:

**Lemma 1.** *Assume that  $0 < x < 1$ , and  $\delta > 0$ . Then for  $\delta \neq 2n\pi$  one has that*

$$\int_{-\delta/2}^{\delta/2} \log(1 - \varepsilon x e^{it}) dt \leq \text{Re} \left( \int_{-\delta/2}^{\delta/2} \log(1 - x e^{i(\theta+t)}) dt \right) \leq \int_{-\delta/2}^{\delta/2} \log(1 + \varepsilon x e^{it}) dx,$$

for  $\varepsilon = \varepsilon(\delta) = \text{sign}(\sin \frac{\delta}{2})$ . When  $\delta = 2n\pi$  the inequalities are in fact equalities for any  $|\varepsilon| = 1$ .

*Proof.* Define

$$F(\theta) = \operatorname{Re} \left( \int_{-\delta/2}^{\delta/2} \log(1 - xe^{i(\theta+t)}) dt \right).$$

By the substitution  $y = \theta + t$  this can be written as

$$F(\theta) = \operatorname{Re} \left( \int_{-\delta/2+\theta}^{\delta/2+\theta} \log(1 - xe^{iy}) dy \right).$$

We will need to determine the extremal values of  $F(\theta)$ . We do this as a calculus exercise. By the fundamental theorem of calculus we get the derivative

$$\begin{aligned} F'(\theta) &= \operatorname{Re} \left( \log(1 - xe^{i(\theta+\delta/2)}) - \log(1 - xe^{i(\theta-\delta/2)}) \right), \\ &= \log \left| \frac{1 - xe^{i(\theta+\delta/2)}}{1 - xe^{i(\theta-\delta/2)}} \right|. \end{aligned}$$

This is zero if and only if

$$(1 - xe^{i\theta+i\delta/2})(1 - xe^{-i\theta-i\delta/2}) = (1 - xe^{i\theta-i\delta/2})(1 - xe^{-i\theta+i\delta/2}).$$

which simplifies to

$$x(e^{i\theta} - e^{-i\theta})(e^{i\delta/2} - e^{-i\delta/2}) = 0,$$

which is true if

$$\sin \theta = 0, \quad x = 0, \quad \text{or} \quad \sin(\delta/2) = 0.$$

We see that  $F'(\theta) = 0$  if  $\delta = 2n\pi$  and hence  $F(\theta)$  is constant in that case, proving the Lemma in case  $\delta = 2n\pi$ . Let us now assume that  $\delta \neq 2n\pi$ . Since  $x \neq 0$ , this means that  $F'(\theta) = 0$  has the solutions

$$\theta = n\pi.$$

We find that

$$\begin{aligned} F''(\theta) &= \operatorname{Re} \left( \left[ \frac{-ixe^{i(\theta+t)}}{1 - xe^{i(\theta+t)}} \right]_{t=-\delta/2}^{t=\delta/2} \right), \\ &= \operatorname{Re} \left( \left[ \frac{-ixe^{i(\theta+t)}(1 - xe^{-i(\theta+t)})}{|1 - xe^{i(\theta+t)}|^2} \right]_{t=-\delta/2}^{t=\delta/2} \right) = \left[ \frac{x \sin(\theta + t)}{|1 - xe^{i(\theta+t)}|^2} \right]_{t=-\delta/2}^{t=\delta/2}. \end{aligned}$$

For  $\theta = 2n\pi$  and  $\theta = (2n+1)\pi$  we find that

$$F''(2n\pi) = 2 \frac{x \sin(\delta/2)}{|1 - xe^{i\delta/2}|^2},$$

and

$$F''((2n+1)\pi) = -2 \frac{x \sin(\delta/2)}{|1 + xe^{i\delta/2}|^2},$$

This shows that  $\theta = 2n\pi$  and  $\theta = (2n+1)\pi$  will be local maxima or minima for  $F(\theta)$  depending on the sign of  $\sin(\delta/2)$ .  $\square$

*Proof of Proposition 1.*

Since its coefficients are completely multiplicative, the Dirichlet series  $A(s)$  and  $\zeta_\delta(s)$  have the Euler-products

$$A(s) = \prod_p (1 - a_p p^{-s})^{-1}, \quad \text{and} \quad \zeta_\delta(s) = \prod_p (1 - \varepsilon_p p^{-s})^{-1}.$$

By the fact that  $\varepsilon_p^2 = 1$  and  $1 - x^2 = (1 - x)(1 + x)$ , we also find that

$$\frac{\zeta(2s)}{\zeta_\delta(s)} = \frac{\prod_p (1 - p^{-2s})^{-1}}{\prod_p (1 - \varepsilon_p p^{-s})^{-1}} = \frac{\prod_p (1 - \varepsilon_p^2 p^{-2s})^{-1}}{\prod_p (1 - \varepsilon_p p^{-s})^{-1}} = \prod_p (1 + \varepsilon_p p^{-s})^{-1}.$$

By taking the logarithm of these products, we obtain

$$\log A(s) = - \sum_p \log(1 - a_p p^{-s}), \quad \log \zeta_\delta(s) = - \sum_p \log(1 - \varepsilon_p p^{-s}),$$

and

$$\log \frac{\zeta(2s)}{\zeta_\delta(s)} = - \sum_p \log(1 + \varepsilon_p p^{-s}).$$

The Proposition follows by using Lemma 1 termwise.  $\square$

As a consequence of Proposition 1, we have the following Lemma:

**Lemma 2.** *We have for  $\sigma \geq 1$  that*

$$\begin{aligned} (i) \quad & \inf_T \int_T^{T+\delta} \log |\zeta(\sigma + it)|^{-1} dt = \int_{-\delta/2}^{\delta/2} \log |\zeta_\delta(\sigma + it)|^{-1} dt, \\ (ii) \quad & \inf_T \int_T^{T+\delta} \log |\zeta(\sigma + it)| dt = \int_{-\delta/2}^{\delta/2} \log \left| \frac{\zeta(2\sigma + 2it)}{\zeta_\delta(\sigma + it)} \right| dt, \end{aligned}$$

where  $\zeta_\delta(s)$  is defined as in Proposition 1.

*Proof.* Since

$$\int_0^\delta \sum_{n=1}^\infty \frac{a_n \Lambda(n)}{n^{\sigma+it} \log n} dt = \sum_{n=1}^\infty \frac{a_n \Lambda(n) (n^{-i\delta} - 1)}{-i(\log n)^2 n^\sigma}$$

is absolutely convergent for any choice of  $|a_n| \leq 1$ ,  $\sigma \geq 1$  and  $\delta > 0$  it follows by using the Euler-product of  $\zeta_\delta(s)$  and  $\zeta(s)$  and integrating the logarithms term wise that it is sufficient to show that there exist some sequence  $T_k$  such that

$$\lim_{k \rightarrow \infty} |p^{iT_k} - \varepsilon_p| = 0, \quad \text{and} \quad \lim_{k \rightarrow \infty} |p^{iT_k} + \varepsilon_p| = 0,$$

respectively for each prime  $p$  in order for

$$\lim_{k \rightarrow \infty} \int_{-\delta/2}^{\delta/2} (\log \zeta(\sigma + iT_k + it) - \log \zeta_\delta(\sigma + it)) dt = 0,$$

and

$$\int_{-\delta/2}^{\delta/2} \left( \log \zeta(\sigma + iT_k + it) - \log \frac{\zeta(2\sigma + 2it)}{\zeta_\delta(\sigma + it)} \right) dt = 0.$$

But this follows by the fact that  $\log p$  are linearly independent over  $\mathbb{Q}$ ; which is equivalent to the fundamental theorem of arithmetic; and Kroenecker's theorem.  $\square$

From this lemma the following analogue of Theorem 3 for the logarithmic  $L^1$ -norm follows.

**Theorem 6.** *We have that*

$$\begin{aligned} (i) \quad & \inf_T \frac{1}{\delta} \int_T^{T+\delta} \log |\zeta(1+it)|^{-1} dt = \log \delta - \log 4 - \gamma + O(\delta^2), \\ (ii) \quad & \inf_T \frac{1}{\delta} \int_T^{T+\delta} \log |\zeta(1+it)| dt = \log \delta - \log 4 + \log \zeta(2) + O(\delta^2). \end{aligned}$$

The lower bound in Theorem 3 will be an immediate consequence:

*Proof.* We will sketch a proof of (i). Theorem 6. (ii) can be proved by the same method, by replacing the use of Lemma 2 (i) with Lemma 2 (ii). We give a full proof of this result by another method later which will yield stronger results (e.g. Proposition 2) also. We use Lemma 2 (i) as a starting point, and hence it is sufficient to calculate

$$\int_{-\delta/2}^{\delta/2} \log |\zeta_\delta(\sigma + it)| dt = \int_{-\delta/2}^{\delta/2} \log |\zeta(\sigma + it)| dt + \int_{-\delta/2}^{\delta/2} \log \left| \frac{\zeta_\delta(\sigma + it)}{\zeta(\sigma + it)} \right| dt, \quad (8)$$

for  $\sigma > 1$ . By using the development of  $\log \zeta(1+it)$  as a power series it follows that

$$\int_{-\delta/2}^{\delta/2} \log |\zeta(1+it)| dt = \delta \log \delta - (\log 2 + 1)\delta + O(\delta^3). \quad (9)$$

By using the Euler products of  $\zeta_\delta(s)$  and  $\zeta(s)$ , taking the logarithms and integrating term wise, we obtain

$$\int_{-\delta/2}^{\delta/2} \log \left| \frac{\zeta_\delta(\sigma + it)}{\zeta(\sigma + it)} \right| dt = \sum_p \int_{-\delta/2}^{\delta/2} \log \left( 1 - \eta \left( \frac{\delta \log p}{4\pi} \right) \right) p^{-\sigma-it} dt,$$

where

$$\eta(x) = \begin{cases} 2, & 1/2 < \{x\} < 1, \\ 0, & 0 \leq \{x\} \leq 1/2. \end{cases}$$

By using the fact that  $\log(1+x) = x + O(x^2)$ , and integrating term wise, this equals

$$\begin{aligned} - \sum_p \int_{-\delta/2}^{\delta/2} \eta\left(\frac{\delta \log p}{4\pi}\right) p^{-\sigma-it} dt + O\left(e^{-1/\delta}\right) = \\ - 4 \sum_p \frac{\sin^-(\delta \log p/2)}{p^\sigma \log p} + O\left(e^{-1/\delta}\right). \end{aligned}$$

where

$$\sin^-(x) = \begin{cases} 0, & \sin(x) \geq 0, \\ -\sin x, & \sin x < 0. \end{cases}$$

By letting  $\sigma \rightarrow 1+$  and using the prime number theorem this equals

$$-4 \int_0^\infty \frac{\sin^-(\delta t/2)}{t^2} dt + O(\delta^3) = -4\delta \int_0^\infty \frac{\sin^-(x/2)}{x^2} dx + O(\delta^3).$$

This integral can be evaluated in terms of Euler's constant and equals

$$\delta(1 - \gamma - \log 2) + O(\delta^3).$$

The result follows from combining this with (8) and (9).  $\square$

*Proof of lower bound in Theorem 3.* This follows from Theorem 6 and Jensen's inequality (7).  $\square$

We also remark that our nonuniversality results for the Riemann zeta-function on the line  $\operatorname{Re}(s) = 1$  can be obtained immediately from Theorem 6, instead of Theorem 3.

Theorem 6 completely solves the problem of obtaining optimal constants in the problem studied in Theorem 3 when the absolute value is replaced by the logarithm of the absolute value of the function.

## 4 Proof of Theorem 3 and Theorem 4

We have proved that  $\zeta_\delta(s)$  is an extremal function for the Logarithmic  $L^1$ -norm. Hence it is natural to ask what this function will give when we consider the  $L^1$ -norm of this function in short intervals. Will it give something better than Examples 1 and Examples 2 for answering Question? The surprising answer is that the absolute value of this function will be approximately constant in short intervals, and hence this will essentially give an extremal example also for Question and will yield a proof of Theorem 3 and Theorem 4. This follows from the following proposition:

**Proposition 2.** *We have uniformly for  $-1 < x < 1$  that*

$$\log \zeta_\delta(1 + ix\delta/2) = -\log \delta + \gamma + \log 4 - ix \left( \frac{1}{2x} \int_0^x \frac{\tan(\pi t/2)}{t} dt + \gamma \right) + O(\delta^2).$$

**Remark 1.** This shows that the absolute value  $|\zeta_\delta(1 + ix\delta/2)|$  is approximately constant when  $-1 < x < 1$ , since the imaginary value of the logarithm will not be relevant when taking absolute values! This is surprising and allows us to prove the upper bounds in Theorems 3 and 4. Further investigation of the function will show that the absolute value of the function will be approximately constant in the intervals  $2n - 1 < x < 2n + 1$  for integers  $n$ , and have discontinuities at  $x = 2n + 1$ .

We first prove the following Lemma:

**Lemma 3.** *We have that*

$$\zeta_\delta(s) = -\log(\delta) + \Theta\left(\frac{s-1}{\delta}\right) + \log(\zeta(s)(s-1)) + O\left((|s|+1)e^{-c_0\delta^{-1/2}}\right),$$

where

$$\Theta(s) = \gamma + \log 4 - \int_0^s \frac{\tanh \pi w}{w} dw,$$

uniformly for  $\operatorname{Re}(s) > 1$  and for some  $c_0 > 0$ .

**Remark 2.** The error term comes from the zero-free region/error term in the prime number theorem of de la Vallée-Poussin [17]. By using the improvements of Vinogradov [51], Korobov [33] and Ford [20] this can be improved. Assuming the Riemann Hypothesis, the error term in Lemma 3 can be improved to  $O(|s|+1)e^{-2\pi/\delta}$ .

*Proof.* We have that

$$\begin{aligned} \log \zeta_\delta(s) &= \log \zeta(s) - \sum_{n=1}^{\infty} \frac{\Lambda(n)\eta\left(\frac{\delta}{4\pi} \log n\right)}{n^s \log n} - \sum_{n \text{ prime power}} \frac{\Lambda(n)\theta\left(\frac{\delta}{4\pi} \log n\right)}{n^s \log n}, \\ &= \log \zeta(s) + \sum_{n=1}^{\infty} \frac{\Lambda(n)\eta\left(\frac{\delta \log n}{4\pi}\right)}{n^s \log n} + O\left(e^{-2\pi/\delta}\right), \quad (\operatorname{Re}(s) \geq 1), \end{aligned}$$

where

$$\eta(x) = \begin{cases} 2, & 1/2 < \{x\} < 1, \\ 0, & 0 \leq \{x\} \leq 1/2. \end{cases} \quad (10)$$

We have that

$$(*) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\eta\left(\frac{\delta}{4\pi} \log n\right)}{n^s \log n} = -\frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\zeta'(s+z)}{\zeta(s+z)} \int_0^\infty \frac{\theta\left(\frac{\delta}{4\pi} \log x\right) x^{z-1} dx}{\log x} dz.$$

By moving the contour to the left of  $\operatorname{Re}(z) = 0$  we will pick up the zeta-function's residue at  $\operatorname{Re}(s + z) = 1$ . From the zero-free region of de la Vallée-Poussin [17] we get an error term and we find that

$$(*) = - \int_0^\infty \frac{\eta\left(\frac{\delta \log x}{4\pi}\right) x^{-s} dx}{\log x} + O\left((|s| + 1)e^{-c_0 \delta^{-1/2}}\right).$$

By using the substitution

$$t = \frac{\delta \log x}{4\pi},$$

we obtain that

$$\begin{aligned} (*) &= - \int_0^\infty \frac{\eta(t) e^{-4\pi(s-1)t/\delta} dt}{t} + O\left((|s| + 1)e^{-c_0 \delta^{-1/2}}\right), \\ &= -\Psi(4\pi(s-1)/\delta) + \log(4\pi(s-1)/\delta) + O\left((|s| + 1)e^{-c_0 \delta^{-1/2}}\right), \end{aligned}$$

where

$$\Psi(s) = -\log s + \int_0^\infty \frac{\eta(t) e^{-st}}{t} dt,$$

and  $\eta(x)$  is defined by (10). It remains to prove that

$$\Theta\left(\frac{s-1}{\delta}\right) = \log 4\pi - \Psi\left(\frac{4\pi(s-1)}{\delta}\right). \quad (11)$$

It is sufficient to prove that their derivatives coincide and that the value at  $\operatorname{Re}(s) = 0$  coincide. We get that

$$\begin{aligned} \Psi'(s) &= - \int_0^\infty \eta(t) e^{-st} dt + \frac{1}{s} = -2 \sum_{n=1}^\infty \int_{n-1/2}^n e^{-st} dt + \frac{1}{s} \\ &= \frac{2}{s} \sum_{n=1}^\infty (e^{-ns} - e^{-(n-1/2)s}) + \frac{1}{s} = \frac{2}{s} \sum_{k=1}^\infty (-1)^k e^{-ks/2} + \frac{1}{s} \\ &= \frac{1}{s} \left( \frac{2}{1 + e^{-ks/2}} - 1 \right) = \frac{1}{s} \left( \frac{2}{e^{s/2} + 1} - \frac{e^{s/2} + 1}{e^{s/2} + 1} \right) \\ &= \frac{1}{s} \cdot \frac{e^{s/2} - 1}{e^{s/2} + 1} = \frac{1}{s} \tanh \frac{s}{4}, \end{aligned}$$

and it is easy to see that (11) is true up to a constant, since the derivatives coincide. To determine the constant we calculate the limit  $\lim_{s \rightarrow 0^+} \Psi(s)$ . We

have when  $0 < s \leq 1/2$  that

$$\begin{aligned}
\Psi(s) &= \int_0^\infty \frac{\eta(x)}{t} e^{-st} dt - \log s, \\
&= 2 \sum_{n=1}^\infty \int_{n-1/2}^n \frac{e^{-st}}{t} dt - \log s, \\
&= (2 + O(s)) \sum_{n=1}^\infty e^{-sn} \int_{n-1/2}^n \frac{1}{t} dt - \log s, \\
&= (2 + O(s)) \sum_{n=1}^\infty e^{-sn} \log \left( 1 + \frac{1}{2n-1} \right) - \log s, \\
&= \sum_{n=1}^\infty \left( 2 \log \left( 1 + \frac{1}{2n-1} \right) - \frac{1}{n} \right) + \sum_{n=1}^\infty \frac{e^{-ns}}{n} - \log s + O(s \log s).
\end{aligned}$$

From the explicit evaluation

$$\sum_{n=1}^\infty \frac{e^{-ns}}{n} = \log(1 - e^{-s}),$$

it follows that

$$\sum_{n=1}^\infty \frac{e^{-ns}}{n} - \log s = O(s), \quad (0 < s \leq 1/2)$$

and we see that

$$\Psi(s) = \sum_{n=1}^\infty \left( 2 \log \left( 1 + \frac{1}{2n-1} \right) - \frac{1}{n} \right) + O(s \log s). \quad (0 < s \leq 1/2) \quad (12)$$

By the well known estimate

$$\sum_{n=1}^N \frac{1}{n} = \log N + \gamma + O(N^{-1}),$$

we obtain

$$\begin{aligned}
\sum_{n=1}^N \left( 2 \log \left( 1 + \frac{1}{2n-1} \right) - \frac{1}{n} \right) &= \\
&= 2 \sum_{n=1}^N \left( \log n - \log \left( n - \frac{1}{2} \right) \right) - \log N - \gamma + O(N^{-1}), \\
&= 2 \left( \log(\Gamma(N+1)) - \log \left( \frac{\Gamma(N+1/2)}{\Gamma(1/2)} \right) \right) - \log N - \gamma + O(N^{-1}), \\
&= 2 \log \left( \frac{\Gamma(N+1)}{\Gamma(N+1/2)} \right) - \log N + 2 \log \Gamma(1/2) - \gamma + O(N^{-1}).
\end{aligned}$$



Stirling's formula

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log(z) - z + \frac{\log(2\pi)}{2} + O(z^{-1}),$$

implies that

$$2 \log \left( \frac{\Gamma(N+1)}{\Gamma(N+1/2)} \right) = \log N + O(N^{-1}).$$

Thus we have that

$$\sum_{n=1}^{\infty} \left( 2 \log \left( 1 + \frac{1}{2n-1} \right) - \frac{1}{n} \right) = -\gamma + 2 \log \Gamma(1/2),$$

and from the fact that  $\Gamma(1/2) = \sqrt{\pi}$ , together with (12) we obtain that

$$\lim_{s \rightarrow 0^+} \Psi(s) = \log \pi - \gamma.$$

□

#### 4.1 Proof of Proposition 2

The Laurent series development of the Riemann zeta-function at  $s = 1$  gives us

$$\log(\zeta(s)(s-1)) = -\gamma(s-1) + O((s-1)^2).$$

By Lemma 3 with  $s = \sigma + ix\delta/2$  and letting  $\sigma \rightarrow 1+$ , this estimate gives us

$$\log(\zeta_{\delta}(1 + ix\delta/2)) = -\log \delta + \log 4 + \gamma - i\gamma x\delta/2 - \int_0^{ix/2} \frac{\tanh \pi w}{w} dw + O(\delta^2),$$

for  $-1 < x < 1$ . With the substitution  $t = iw/2$  we obtain

$$\log(\zeta_{\delta}(1 + ix)) = -\log \delta + \gamma + \log 4 + i \left( -\gamma x - \frac{1}{2} \int_0^x \frac{\tan \pi t/2}{t} dt \right) + O(\delta^2).$$

□

#### 4.2 Proof of Theorem 3 and 4

Since the imaginary part of the logarithm can be disregarded (see Remark 2), when taking absolute values, we see that Proposition 2 gives us

**Lemma 4.** *We have that uniformly for  $-1 < x < 1$  that*

$$\begin{aligned} (i) \quad & \left| \frac{\zeta(2(1 + ix\delta/2))}{\zeta_{\delta}(1 + ix\delta/2)} \right| = \frac{\delta \pi^2 e^{-\gamma}}{24} + O(\delta^3), \\ (ii) \quad & |\zeta_{\delta}(1 + ix\delta)|^{-1} = \frac{\delta e^{-\gamma}}{4} + O(\delta^3). \end{aligned}$$

*Proof.* Part (ii) is an immediate consequence of Proposition 2. Part (i) follows from the Taylor expansion at  $s = 2$  for  $\zeta(s)$  since we have for real valued  $x$  that

$$\begin{aligned} |\zeta(2 + 2ix)| &= |\zeta(2) + \zeta'(2)ix + O(x^2)|, \\ &= \sqrt{(\zeta(2) + O(x^2))^2 + O(x^2)}, \\ &= \zeta(2) + O(x^2). \end{aligned}$$

□

*Proof of Theorem 4.* The lower bound in Theorem 4 follows from Jensen's inequality and Lemma 2. The upper bound follows in the same way as Lemma 2. Given  $\epsilon > 0$  and  $\sigma > 1$  there exists some  $T$  such that

$$\max_{t \in [-\delta/2, \delta/2]} |\zeta(\sigma + iT + it) - \zeta_\delta(\sigma + it)| < \epsilon. \quad (13)$$

By Lemma 4 (ii) and the triangle inequality Theorem 4 (ii) follows. Theorem 4 (i) follows in a similar way. □

The proof of Theorem 3 is somewhat more complicated since (13) can not be proved by absolute convergence when  $\sigma = 1$ . We will apply methods coming from the theory of Universality of  $L$ -functions. We first state a Lemma in slightly more generality than we presently need for later purposes.

**Lemma 5.** *Let for some  $\sigma_1 < 1$  and  $T_0 > 0$*

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

*be a Dirichlet series that is analytic for  $\operatorname{Re}(s) > \sigma_1$ ,  $|\operatorname{Im}(s)| > T_0$ , absolutely convergent for  $\operatorname{Re}(s) > 1$  and fulfill the mean square property*

$$\sup_{\sigma > \sigma_1} \int_{T_0}^T |A(\sigma + it)|^2 dt \ll T. \quad (14)$$

*Let  $\delta > 0$  and*

$$B(s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

*such that  $a_n/b_n$  is a unimodular completely multiplicative function and such that  $B(1 + it) = \lim_{\sigma \rightarrow 1+} B(\sigma + it)$  is bounded and continuous on  $[-\delta/2, \delta/2]$ . Then there exists for any  $\epsilon > 0$  a real number  $T$  such that*

$$\max_{t \in [-\delta/2, \delta/2]} |B(1 + it) - A(1 + it + iT)| < \epsilon.$$

*Proof.* This follows from the theory of Universality of  $L$ -functions, and is a variant of e.g. Steuding [47, Theorem 4.12] a result due to Laurinćikas [36, 37]. Note that this is formulated in a slightly different way, and the conditions for the theorem to hold are somewhat different. The same proof method still applies though. □

*Proof of Theorem 3.* The lower bound in Theorem 3 follows immediately from Lemma 2. By applying Lemma 5 on  $A(s) = \zeta(s)^{-1}$ , and  $B(s) = \zeta_{\delta+\epsilon}(s)^{-1}$  we find that

$$\max_{t \in [-\delta/2, \delta/2]} |\zeta(1 + iT + it)^{-1} - \zeta_{\delta+\epsilon}(1 + it)^{-1}| < \epsilon.$$

Combining this with the triangle inequality and Lemma 4 (ii) this implies that

$$\frac{\delta e^{-\gamma}}{4} + O(\delta^3) - \epsilon < |\zeta(1 + iT + it)^{-1}| < \frac{\delta e^{-\gamma}}{4} + O(\delta^3) + \epsilon.$$

The upper bound in Theorem 3 (ii) then follows by choosing  $0 < \epsilon < \delta^3$ . The lower bound follows immediately from Lemma 2.

The upper bound in Theorem 3 (i) follows in the same way by Lemma 4 (ii) by choosing  $A(s) = \zeta(s)$  and  $B(s) = \zeta(2s)/\zeta_{\delta+\epsilon}(s)$  in Lemma 5.  $\square$

### 4.3 The $L^p$ -norm case

**Theorem 7.** *We have the following estimates for the  $L^p$  norm, for  $p > 0$  of the zeta-function and its inverse in short intervals:*

$$\begin{aligned} (i) \quad & \inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |\zeta(1 + it)|^p dt \right)^{1/p} = \frac{\pi^2 e^{-\gamma}}{24} \delta + O(\delta^3), \\ (ii) \quad & \inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |\zeta(1 + it)|^{-p} dt \right)^{1/p} = \frac{e^{-\gamma}}{4} \delta + O(\delta^3), \end{aligned}$$

for  $\delta > 0$ . Furthermore, both estimates are valid if  $\inf_T$  is replaced by  $\liminf_{T \rightarrow \infty}$ , and if  $1 + it$  is replaced by  $\sigma + it$  and the infimum is also taken over  $\sigma > 1$ .

*Proof.* This result follows in the same way from Proposition 2, Lemma 2, Lemma 4 and Lemma 5 as Theorem 3 and Theorem 4.  $\square$

### 4.4 Sup-norm case

We can also state our result in sup-norm case. This might in fact be the nicest formulation of our result.

**Theorem 8.** *We have that*

$$\inf_T \max_{t \in [T, T+\delta]} |\zeta(1 + it)| = \frac{e^{-\gamma} \pi^2}{24} \delta + O(\delta^3),$$

and

$$\sup_T \min_{t \in [T, T+\delta]} |\zeta(1 + it)| = \frac{4e^{-\gamma}}{\delta} + O(\delta).$$

*Proof.* This result follows in the same way from Proposition 2, Lemma 2, Lemma 4 and Lemma 5 as Theorem 3.  $\square$

## 5 General Dirichlet series with an Euler product

We will now show how we can obtain similar results for other Dirichlet series than the Riemann zeta-function. In particular we have Dirichlet  $L$ -functions and Rankin-Selberg  $L$ -functions in mind. Hecke  $L$ -functions of cusp forms will be somewhat more complicated and we will show somewhat weaker results for that case.

### 5.1 Multiplicative arithmetical functions

First we will state a rather general theorem that is valid for Dirichlet series with multiplicative coefficients. Later we will specialise it to the case of completely multiplicative functions and functions with positive coefficients

**Theorem 9.** *Let  $A(s)$  be a Dirichlet series with multiplicative coefficients*

$$A(s) = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{p \text{ prime}} f_p(p^{-s}),$$

where

$$f_p(z) = 1 + \sum_{k=1}^{\infty} a(p^k)z^k,$$

such that  $A(s)$  is absolutely convergent for  $\operatorname{Re}(s) > 1$  and

$$\begin{aligned} \sum_{k=2}^{\infty} \sum_{\substack{p \text{ prime} \\ p^k \geq N}} \frac{|a(p^k)|}{p^k} &\ll (\log \log N)^{-2}, \\ \sum_{\substack{p \text{ prime} \\ p < N}} \frac{|a(p)|}{p} &= \alpha \log \log N + \beta + O((\log \log N)^{-2}). \end{aligned}$$

Then

$$\lambda_1 = \sum_p \left( |a_p|p^{-1} - \max_{|z|=1/p} \log |f_p(z)| \right),$$

is convergent and we have for each  $0 < p < \alpha$  that

$$\inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |A(1+it)|^{-p} dt \right)^{1/p} = \frac{e^{-\gamma+\beta-\lambda_1}}{4} \delta^\alpha (1 + O(\delta^2)).$$

If furthermore the local Euler-factors  $f_p(z)$  have no zeroes for  $|z| = 1/p$ . Then

$$\lambda_0 = \sum_p \left( \min_{|z|=1/p} \log |f_p(z)| + |a_p|p^{-1} \right),$$

is convergent and we have for each  $0 < p < 1/\alpha$  that

$$\inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |A(1+it)|^p dt \right)^{1/p} = \frac{e^{-\gamma+\beta-\lambda_0}}{4} \delta^\alpha (1 + O(\delta^2)).$$

Furthermore if the error terms  $(\log \log N)^{-2}$  are replaced by  $o(1)$ , the theorem is still true if we replace the error terms  $O(\delta^2)$  by  $o(1)$ .

*Proof.* This follows by the same proof method as used to prove Theorem 4.  $\square$

**Theorem 10.** Suppose that  $A(s)$  fulfill all conditions of Theorem 9 and furthermore that  $A(s)$  fulfill the mean square property (14) for some  $\sigma_1 < 1$  and is analytic for  $\sigma_1 < \operatorname{Re}(s)$ ,  $T_0 < |\operatorname{Im}(t)|$ . Then Theorem 9 is true for all  $p > 0$ . Furthermore we have the corresponding result in sup-norm:

$$\inf_T \max_{t \in [T, T+\delta]} |A(1+it)|^{-1} = \frac{e^{-\gamma+\beta-\lambda_1}}{4} \delta^\alpha (1 + O(\delta^2)).$$

If the local Euler-factors  $f_p(z)$  have no zeroes for  $|z| = 1/p$  then

$$\inf_T \max_{t \in [T, T+\delta]} |A(1+it)| = \frac{e^{-\gamma+\beta-\lambda_0}}{4} \delta^\alpha (1 + O(\delta^2)).$$

*Proof.* The proof uses Lemma 5 in the same way as the proof of Theorem 3.  $\square$

## 5.2 Completely multiplicative arithmetical functions

In the case of completely multiplicative functions we can use the result previously proved, since if  $a(n)$  is completely multiplicative we have that

$$f_p(z) = \sum_{k=1}^{\infty} a(p^k) z^k = \sum_{k=1}^{\infty} (a(p)z)^k = \frac{1}{1 - a(p)z}.$$

It is clear that

$$\min_{|z|=1} \log |f_p(z)| = -\log(1 + |a(p)|/p) \quad \text{and} \quad \max_{|z|=1} \log |f_p(z)| = -\log(1 - |a(p)|/p).$$

In this case Theorem 9 becomes

**Theorem 11.** Suppose that

$$A(s) = \sum_{n=1}^{\infty} a(n) n^{-s},$$

is a Dirichlet series absolutely convergent for  $\operatorname{Re}(s) > 1$  such that  $a(n)$  is a completely multiplicative function where  $|a(n)| = n$  if and only if  $n = 1$ . Suppose that

$$\sum_{n=1}^N \frac{\Lambda(n) |a(n)|}{n \log n} = \alpha \log \log N + \beta + O((\log \log N)^{-2}).$$

Then for any  $0 < p < 1/\alpha$

$$\inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |A(1+it)|^p dt \right)^{1/p} = \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^2} \cdot \frac{e^{-\beta}}{4} \delta^\alpha (1 + O(\delta^2)),$$

and

$$\inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |A(1+it)|^{-p} dt \right)^{1/p} = \frac{e^{-\beta}}{4} \delta^\alpha (1 + O(\delta^2)).$$

Furthermore if the error terms  $(\log \log N)^{-2}$  is replaced by  $o(1)$ , the theorem is still true if we replace the error terms  $O(\delta^2)$  by  $o(1)$ .

In the completely multiplicative case Theorem 10 becomes

**Theorem 12.** Suppose that  $A(s)$  fulfill all conditions of Theorem 11 and furthermore that  $A(s)$  fulfill the mean square property (14) for some  $\sigma_1 < 1$  and is analytic for  $\sigma_1 < \operatorname{Re}(s)$ ,  $T_0 < |\operatorname{Im}(t)|$ . Then the results of Theorem 11 are true for any  $p > 0$ . Furthermore we have the corresponding results in sup-norm.

$$\begin{aligned} \inf_T \max_{t \in [T, T+\delta]} |A(1+it)| &= \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^2} \cdot \frac{e^{-\beta}}{4} \delta^\alpha (1 + O(\delta^2)), \\ \sup_T \min_{t \in [T, T+\delta]} |A(1+it)| &= 4e^\beta \delta^{-\alpha} (1 + O(\delta^2)). \end{aligned}$$

### 5.2.1 The Riemann zeta-function revisited

If  $a(n) = 1$ , we see that  $A(s) = \zeta(s)$  is the Riemann zeta-function and we recover Theorems 3, 4, 7 and 8.

### 5.2.2 Dirichlet L-series

Since the Dirichlet L-series also have completely multiplicative coefficients we can use Theorem 10, 11 and 12 to obtain a version of Theorem 7 and Theorem 8 (which includes Theorem 3 and 4 as special cases):

**Theorem 13.** Let  $L(s, \chi)$  be an Dirichlet L-series, where  $\chi$  is a character mod  $D$ . Then for  $p > 0$  we have the following estimates:

$$\begin{aligned} (i) \quad \inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |L(1+it, \chi)|^p dt \right)^{1/p} &= \frac{\pi^2 e^{-\gamma} D}{24\Phi(D)} \delta + O(\delta^3), \\ (ii) \quad \inf_T \left( \int_T^{T+\delta} |L(1+it, \chi)|^{-p} dt \right)^{1/p} &= \frac{e^{-\gamma} D}{4\phi(D)} \delta + O(\delta^3), \end{aligned}$$

for  $\delta > 0$ . Furthermore, the results are true if the  $L^p$ -norm is replaced by the sup-norm and both estimates are valid if  $\inf_T$  is replaced by  $\liminf_{T \rightarrow \infty}$ , and if  $1+it$  is replaced by  $\sigma+it$  and the infimum is also taken over  $\sigma > 1$ .

### 5.3 Functions with positive coefficients

In the case of positive coefficients  $a(n)$ , it will be easier to treat

$$f_p(z) = \sum_{k=0}^{\infty} a(p^k) z^k, \quad (15)$$

since it is clear that

$$\max_{|z|=1} |f_p(z)| = f_p(1). \quad (16)$$

The minimum is somewhat more complicated. It is not so difficult to see that  $z = -1$  is a local minimum of  $f_p(z)$  under some minimal assumptions, but it is not a global minimum in general. Thus we only treat the case with the maximum.

**Theorem 14.** *Suppose that*

$$A(s) = \sum_{n=1}^{\infty} a(n) n^{-s}, \quad (a)$$

*is a Dirichlet series absolutely convergent for  $\operatorname{Re}(s) > 1$  such that  $a(n)$  is a positive multiplicative function, and*

$$\sum_{k=2}^{\infty} \sum_{\substack{p \text{ prime} \\ p^k \geq N}} \frac{|a(p^k)|}{p^k} \ll (\log \log N)^{-2},$$

$$\sum_{n=1}^{\infty} \frac{\Lambda(n) a(n)}{n \log n} = \alpha \log \log N + \beta + O((\log \log N)^{-2}).$$

*Then*

$$\inf_T \left( \int_T^{T+\delta} |A(1+it)|^{-p} dt \right)^{1/p} = \frac{e^{-\beta}}{4} \delta^{\alpha} (1 + O(\delta^2)).$$

*Furthermore if the error terms  $(\log \log N)^{-2}$  are replaced by  $o(1)$ , the theorem is still true if we replace the error term  $O(\delta^2)$  by  $o(1)$ .*

**Theorem 15.** *Suppose that  $A(s)$  fulfill all conditions of Theorem 14 and furthermore that  $A(s)$  fulfill the mean square property (14) for some  $\sigma_1 < 1$  and is analytic for  $\sigma_1 < \operatorname{Re}(s)$ ,  $T_0 < |\operatorname{Im}(t)|$ . Then Theorem 14 is true also for  $p \geq \alpha$  and with  $L^p$ -norm replaced by sup-norm.*

#### 5.3.1 The Rankin-Selberg $L$ -function

Let  $f(z)$ , and  $g(z)$  be automorphic forms on the full modular group, with Fourier coefficients  $a(n)$  and  $b(n)$  respectively. The Rankin-Selberg  $L$ -function is defined by

$$L(s, f \times g) = \zeta(2s) L(s, f \otimes g),$$

where

$$L(s, f \otimes g) = \sum_{n=1}^{\infty} a(n)b(n)n^{-s},$$

denotes the convolution Rankin-Selberg  $L$ -function. If  $g(z) = f(z)$  is a Hecke Eigenform we get that

$$L(s, f \times f) = \zeta(2s)L(s, f \otimes f) = \zeta(s) \sum_{n=1}^{\infty} t(n)^2 n^{-s}. \quad (17)$$

We have that  $t(n)$  is real (see e.g. [40]) and hence both the convolution Rankin-Selberg  $L$ -function as well as the Rankin-Selberg zeta function have positive coefficients, since  $t(n)^2$  will be positive and  $\zeta(2s)$  and  $\zeta(s)/\zeta(2s)$  have positive coefficients. Rankin [43] and Selberg [44] proved that  $L(s, f \times f)$  is a holomorphic function in the neighborhood of  $\operatorname{Re}(s) = 1$  with the exception of a pole of order 1 and residue  $1/12\pi$ . The corresponding result for Maass wave-forms can be found in [40]. It is also well-known that the mean square property is true for the Rankin-Selberg zeta-function in the critical strip sufficiently close to  $\operatorname{Re}(s) = 1$ . This means that we can apply Theorem 14 and we obtain the theorems:

**Theorem 16.** *Let  $L(s, f \times \bar{f})$  be a Rankin-Selberg  $L$ -function defined by (17), and where  $f$  is a Hecke-Eigen-form. Then*

$$\inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |L(1+it, f \times \bar{f})|^{-p} dt \right)^{1/p} = \frac{3\pi e^{-\gamma}}{\delta} + O(\delta^3).$$

*We also have that there exist a constant  $C$  depending on  $f$ , but not on  $\delta$  which can be calculated by Theorem 10, such that*

$$\inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |L(1+it, f \times \bar{f})|^p dt \right)^{1/p} = C\delta + O(\delta^3)$$

*The corresponding result are also true when the  $L^p$ -norm is replaced by the sup-norm.*

*Proof.* The needed results to apply Theorem 10 and Theorem 15 are well-known for the Rankin-Selberg  $L$ -function. The constant comes from the residue of the Rankin-Selberg  $L$ -function at  $s = 1$ .  $\square$

**Theorem 17.** *Let  $L(s, f \otimes \bar{f})$  be a convolution Rankin-Selberg  $L$ -function defined by (17), and where  $f$  is a Hecke-Maass cusp form. Then*

$$\inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |L(1+it, f \otimes \bar{f})|^{-p} dt \right)^{1/p} = \frac{\pi^3 e^{-\gamma}}{2} \delta + O(\delta^3).$$

*We also have that there exist a constant  $C$  depending on  $f$ , but not on  $\delta$  which can be calculated by Theorem 10, such that*

$$\inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |L(1+it, f \otimes \bar{f})|^p dt \right)^{1/p} = C\delta + O(\delta^3)$$



*Proof.* The needed results to apply Theorem 10 and Theorem 15 are well-known for the convolution  $L$ -function. It can also be seen from Theorem 16 and Eq. (17).  $\square$

**Problem 2.** Calculate the constants  $C$  in Theorems 16 and 17.

The results in this section are proven for  $GL(2)$   $L$ -functions. However similar results follow for  $GL(3)$  and  $GL(4)$   $L$ -functions as well, by results of Kim [30] and Kim and Shahidi [31,32]. We will not do this in this paper.

### 5.3.2 Higher order convolution $L$ -functions

Let as in the previous section  $a(n)$  be the Fourier coefficients of a  $GL(2)$   $L$ -function. The higher order convolution  $L$ -function is defined as follows

$$L(s, \otimes^{2k} f) = \sum_{n=1}^{\infty} a(n)^{2k} n^{-s}. \quad (18)$$

From the results of Kim-Shahidi [31,32] and Kim [30] for the symmetric  $n$ -th power  $L$ -functions for  $n = 1, 2, 3, 4, 5, 6, 7, 8$  it follows that

$$L(s, \otimes^{2k} f) = \sum_{n=1}^{\infty} a(n)^{2k} n^{-s} \sim C_k (\log s)^{C(k)} (1 + O(\log s)^{-1}), \quad (19)$$

for  $k = 1, 2, 3, 4$ , where

$$C(k) = \frac{(2k)!}{(k+1)!k!} \quad (20)$$

denote the Catalan numbers. Similarly it follows that if  $f$  is a holomorphic cusp form that (19) is true for all  $k \geq 1$ , by the recent result of Taylor et. al. [9] who proved that the Symmetric  $n$ -th power  $L$ -functions are holomorphic and nonvanishing for  $\text{Re}(s) \geq 1$  which already Serre [46] had showed implied the Sato-Tate conjecture.

**Theorem 18.** *Let  $f$  be a Hecke-Maass eigenform for the full modular group. Then there exists constants  $A_k$  and  $B_k$  for  $k = 1, 2, 3, 4$ , and for any  $k \geq 2$  integer if  $f$  is a holomorphic form such that for any  $1/C(k) > p > 0$  we have that*

$$\inf_T \left( \int_T^{T+\delta} |L(s, \otimes^k f)|^{-p} dt \right)^{1/p} = A_k \delta^{C(k)} (1 + O(\delta^2)).$$

*We also have that there exist a constant  $C$  depending on  $f$ , but not on  $\delta$  which can be calculated by Theorem 9,10, such that*

$$\inf_T \left( \int_T^{T+\delta} |L(s, \otimes^k f)|^p dt \right)^{1/p} = B_k \delta^{C(k)} (1 + O(\delta^2)).$$

*Furthermore for  $k = 1, 2, 3, 4$  we can choose any  $p > 0$  and the corresponding result for sup-norm is also true.*

*Proof.* The result follows from Theorems 9,10 and 14,15 and by (19) and some Tauberian argument. For the Maass-Wave-form case, it follows from Kim-Shahidi's result that the required mean-square property is true.  $\square$

## 6 A lower bound for more general Dirichlet series

We remark that what we needed in the proof of the lower bound in Theorems 3 and 4 was the following estimate

$$\sum_{n=1}^N \frac{\Lambda(n)}{n \log n} = \log \log N + O(1),$$

as well as Jensen's inequality. Hence the prime number theorem suffices. Similarly the same proof method can be used to estimate Dirichlet series when similar estimates for the coefficients of the logarithm of the Dirichlet series applies.

**Theorem 19.** *If*

$$\log A(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

*and*

$$\sum_{n=1}^N \frac{|a_n|}{n} \leq \alpha \log \log N + O(1), \quad (0 < \alpha < 1).$$

*Then*

$$\delta^{1+\alpha} \ll \int_T^{T+\delta} |A(1+it)| dt \ll \delta^{1-\alpha}, \quad 0 < \delta \leq 1.$$

*Proof.* It follows that

$$-\alpha \log \delta + O(1) \leq \frac{1}{\delta} \int_T^{T+\delta} \log |A(1+it)| dt \leq \alpha \log \delta + O(1)$$

in a similar way as the proof of Theorem 6. The conclusion follows from Jensen's inequality.  $\square$

### 6.1 $L^p$ estimates of the Riemann zeta-function

Although Theorem 3 gives a stronger result for the lower bound we obtain the following consequence of Theorem 19 when applied on the Riemann zeta-function:

**Corollary 1.**

$$\delta^{|p|} \ll \frac{1}{\delta} \int_T^{T+\delta} |\zeta(1+it)|^p dt \ll \delta^{-|p|} \quad (0 < |p|, \delta < 1)$$

We remark that the upper bound shows that  $\zeta^\theta(s)$  for  $|\theta| < 1/2$  belongs to the Hardy class  $H^2$  of Dirichlet series, see [7, remark 4].

## 6.2 Maass wave forms and a result of Holowinsky

We have the following results for Hecke L-series of Hecke-Maass cusp forms:

**Corollary 2.** *Let  $H(s)$  be a Hecke L-series attached to a Maass wave form or holomorphic cusp form. Then*

$$\delta^{23/12} \ll \int_T^{T+\delta} |H(1+it)| dt \ll \delta^{1/12}, \quad (0 < \delta \leq 1)$$

*Proof.* This follows from a recent results of Holowinsky. Let

$$H(s) = \sum_{n=1}^{\infty} a(n)n^{-s}.$$

From Lemma 4.1 in [26] it follows that

$$\sum_{p \leq X} \frac{|a(p)|}{p} \leq 11/12 \log \log X + O(1).$$

Corollary 2 is now a consequence of Theorem 19.  $\square$

Holowinsky's result was proved by using recent important results of Symmetric  $n$ -th power  $L$ -functions of Kim and Shahidi [31] for  $n = 1, \dots, 8$  (Just the cases  $n = 2, 4, 6$  are in fact used by Holowinsky), and was an important ingredient in his and Soundararajan's proof of Quantum unique ergodicity [27]. We remark that a previous result of Elliot-Moreno-Shahidi [19] proves this with the somewhat weaker bounds  $\delta^{35/18}$  and  $\delta^{1/18}$  under the assumption of the Ramanujan-Petersson conjecture, and hence by Deligne [18] for holomorphic cusp forms. Also in the case of Holomorphic cusp forms improvements along these lines have been done by Tenenbaum [49] and Wu [56], although these results are superseded by the latest results on Sato-Tate of Taylor et.al.

## 6.3 Holomorphic cusp forms and the Sato-Tate conjecture

In the case of holomorphic cusp forms the results of Holowinsky can be improved. In their recent work Tom Barnet-Lamb, David Geraghty, Michael Harris and Richard Taylor [9] prove the Sato-Tate conjecture for all holomorphic newforms of weight  $k \geq 2$  on the group  $\Gamma_0(N)$ . By the Shimura-Taniyama conjecture, proved By Wiles [52] for square-free  $q$  and completely settled by Breuil-Conrad-Diamond-Taylor [15] an  $L$ -function of an elliptic curve is a cusp newform of weight 2 for  $\Gamma_0(q)$ . Thus this result properly extend some of their previous results on the Sato-Tate for  $L$ -functions associated with Elliptic curves with non integral  $j$ -invariant [16, 25, 48]. Let us now assume that

$$f(z) = \sum_{n=1}^{\infty} a(n)n^{(k-1)/2}e(nz) \tag{21}$$

is a holomorphic new form for  $\Gamma_0(N)$  of weight  $k \geq 2$ . We remark that in particular this includes all holomorphic cusp forms for the full modular group

since all these forms have weight  $k \geq 12$ . The Sato-Tate conjecture, now a theorem states that  $a(p)/2$  should be equidistributed in  $[-1, 1]$  with respect to the measure  $2/\pi\sqrt{1-t^2}$ . We obtain that (This is essentially the calculation in [19, p.511]):

$$\sum_{p \leq N} \frac{|a(p)|}{p} \sim 2 \times \frac{2}{\pi} \int_{-1}^1 |t| \sqrt{1-t^2} dt \times \log \log N \sim \frac{8}{3\pi} \log \log N.$$

This result and Theorem 19 allows us to improve Corollary 2 in the case of holomorphic cusp forms. Let

$$H(s) = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_p (1 - a(p)p^{-s} + p^{-2s})^{-1} \quad (22)$$

be the Hecke L-series that corresponds to the cusp form  $f(z)$  defined by Eq. (21). Then we have the following Corollary

**Corollary 3.** *Let  $H(s)$  be the Hecke L-series attached to a holomorphic newform of weight  $\geq 2$  (Defined by Eqs. (21) and (22)). Then*

$$\delta^{1+8/(3\pi)+\epsilon} \ll_{\epsilon} \int_T^{T+\delta} |H(1+it)| dt \ll_{\epsilon} \delta^{1-8/(3\pi)-\epsilon}, \quad (0 < \delta \leq 1) \quad (23)$$

Furthermore

$$\liminf_{T \rightarrow \infty} \int_T^{T+\delta} |H(1+it)| dt \ll_{\epsilon} \delta^{1+8/(3\pi)-\epsilon}.$$

We remark that  $8/(3\pi) = 0.848826\dots$ . Elliot-Moreno-Shahidi [19] proves related results under a somewhat sharper variant of the Sato-Tate conjecture that would allow us to remove the  $\epsilon$  in Corollary 3. What they proved under this somewhat sharper version of Sato-Tate is the following

$$\sum_{p \leq N} \frac{|a_p|}{p} = \frac{8}{3\pi} \log \log N + O(1). \quad (24)$$

This would follow if we would have a sufficiently good estimates (explicit in  $n$ ) for the symmetric  $n$ 'th power  $L$ -functions close to  $\text{Re}(s) = 1$ . We remark that a conjecture of Akiyama-Tanigawa [1] proposed for elliptic curves  $L$ -functions (but it should be possibly to state for other  $L$ -functions as well) would also imply this, and in fact would under a general version of the Riemann hypothesis yield the much stronger error term  $O(N^{-1/2+\epsilon})$  in Eq. (24). For a discussion about this, see the survey of Mazur [39]. We therefore suggest the following problem:

**Problem 3.** Prove that it is possible to remove  $\epsilon$  in Corollary 3.

The error term  $O(\log \log N)^{-2}$  in (24) would by Theorem 10 yield even sharper results. In fact we believe the following to be true (in particular it would follow from Akiyama-Tanigawa's conjecture):

**Conjecture.** Let  $a_n$  be the Fourier coefficients for a modular form. Then

$$\sum_{p \leq N} \frac{|a_p|}{p} = \frac{8}{3\pi} \log \log N + B + O(\log \log N)^{-2}.$$

for some constant  $B$  (depending on the form).

Then it follows that

**Theorem 20.** *Let  $a(n)$  be the Fourier coefficients for a modular form such that Conjecture is true. Then there exists some constants  $C_1, C_2$  such that*

$$\inf_T \max_{t \in [T, T+\delta]} |H(1+it)| = C_1 \delta^{8/(3\pi)} (1 + O(\delta^2)),$$

$$\sup_T \min_{t \in [T, T+\delta]} |H(1+it)| = C_2 \delta^{-8/(3\pi)} (1 + O(\delta^2)).$$

*The corresponding result for  $L^p$  norm would also be true.*

*Proof.* This follows from Theorem 10. □

## 7 Hilbert modular forms

Recently Barnet-Lamb, Geraghty and Toby [10] proved the Sato-Tate conjecture for Hilbert modular forms by the same potential automorphy argument used for classical modular forms. Although we are not stating them here, by similar reasoning as in this paper analogues for Theorems 18, Theorem 20 and Corollary 3 can be found for Hecke  $L$ -series of Hilbert modular forms.

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